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THE COTYPE CONSTANT AND AN ALMOST EUCLIDEAN DECOMPOSITION FOR FINITE-DIMENSIONAL NORMED SPACES

BY

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ABSTRACT

Every $2n$-dimensional normed space E contains two n-dimensional subspaces E_1 and E_2 which are orthogonal with respect to the inner product induced by the John ellipsoid of E and which satisfy $d(E_i, l_2^n) \leqq f(K_2(E))$, where $f(K_2(E))$ is some number that depends only on the cotype constant of E, denoted $K_2(E)$.

1. Introduction

In [6] B. Kashin proved that each space l_1^{2n} can be expressed as a direct sum of two n-dimensional subspaces which are orthogonal with respect to the inner product in l_2^{2n} and whose Banach–Mazur distance from Euclidean space is less than some absolute constant. In [18] the second-named author showed that every finite-dimensional normed space admits such a so-called Kashin decomposition if a certain affine invariant associated with the space, known as the volume ratio, is small. The authors of [19] went on to show that the volume ratios of the members of many infinite families of normed spaces (e.g. $l_2^n \hat{\otimes} l_2^n$) are uniformly bounded, thus proving that these normed spaces admit a Kashin decomposition. The problem of calculating the volume ratio for certain families of normed spaces has been addressed by various authors (e.g. [14], [16]), and a systematic investigation of the volume ratio is carried out in [12].

Many of the families of normed spaces which have a uniformly bounded volume ratio also have a uniformly bounded cotype constant. The main result of this article shows that the latter property alone guarantees the existence of a Kashin decomposition.

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Some remarks are required to describe the organization of this article. The results in §3 on the relation between the volume ratio of a normed space and the type constant of its dual, though not new, are nevertheless not widely known. Since they are of considerable interest in their own right and since they play a crucial role in the arguments of §4, proofs of these results are supplied. The new results appear in §4, where it is shown that, for each θ in $(0,1)$, the Banach–Mazur distance from Euclidean space of most (i.e. of a set of large measure) $[\theta n]$-dimensional subspaces of an n-dimensional normed space E is bounded above by a constant depending only on θ and the cotype constant of E. The Kashin decomposition theorem stated in the abstract is deduced from the latter result. Finally, §5 contains the proof of Lemma 1, which is used in §4, and some remarks and open problems.

2. Notation and terminology

The Euclidean space l_2^n consists of all real n-tuples $x = (x_1, \dots, x_n)$ with the norm $\|x\| = (\Sigma_1^n |x_i|^2)^{1/2}$; the unit ball of l_2^n is denoted b_2^n, the unit sphere S^{n-1}, and the rotationally invariant probability measure on S^{n-1} is denoted σ_{n-1}. (G_k^n, μ_k^n) denotes the Grasmannian manifold of all k-dimensional subspaces of l_2^n equipped with its natural probability measure induced by Haar measure on the orthogonal group. If $1 \leqq k \leqq l \leqq n$ and if F is any l-dimensional subspace of l_2^n, let the Grasmannian manifold of all k-dimensional subspaces of F and its natural probability measure be denoted by (G_k^F, μ_k^F). Suppose that f is any integrable function on G_k^n; it follows from the invariance of Haar measure that

$$\int f d\mu_k^n = \int \left(\int f d\mu_k^F \right) d\mu_l^n.$$

In particular if $f = \chi_B$, then

$$\mu_k^n(B) = \int \mu_k^F(B \cap G_k^F) d\mu_l^n(F), \tag{0}$$

a formula which will be of assistance in §4; we will also comment on various "isoperimetric" problems related to (0) at the very end of this paper.

Suppose now that E is an n-dimensional normed space. The Banach–Mazur distance coefficient, denoted $d(E, l_2^n)$, is defined by

$$d(E, l_2^n) = \inf\{\|T\| \, \|T^{-1}\| : T : E \to l_2^n \text{ is an isomorphism}\}.$$

It will be convenient to identify E with l_2^n: let B_E denote the unit ball of E, and let $\mathscr{E}$ denote the symmetric ellipsoid which is wholly contained in B_E and is of maximum volume (the so-called John ellipsoid). Then the volume ratio of E, denoted $\mathrm{vr}(E)$, is defined by

$$\mathrm{vr}(E)=\left(\frac{\mathrm{vol}_n(B_E)}{\mathrm{vol}_n(\mathscr{E})}\right)^{1/n},$$

where $\mathrm{vol}_n(\cdot)$ denotes Lebesgue measure in $\mathbf{R}^n$.

Suppose that H_1 is a finite-dimensional Hilbert space and that X is a normed space. If γ denotes the canonical Gaussian probability measure on H_1, then an operator $T\in L(H_1,X)$ (the collection of all operators from H_1 into X) can be given the norm

$$l(T)=\left(\int_{H_1}\|Tx\|^2\,d\gamma\right)^{1/2}.$$

If H_2 is any other finite-dimensional Hilbert space and $S:H_2\to H_1$ is a linear map, then it follows that $l(TS)\leqq\|S\|\,l(T)$. Use the fact that γ is invariant under unitary transformations and that any norm one operator is a convex combination of unitary operations.

We also require the notion of a 2-absolutely summing operator. Suppose that X and Y are Banach spaces and that $T:X\to Y$ is a bounded operator. Then T is said to be 2-absolutely summing if there exists a constant $C>0$ such that for each $\varepsilon>0$ there exist a probability space (Ω,Σ,μ) and the following factorisation of T in which $\|u\|\,\|v\|\leqq C+\varepsilon$:

$$X\xrightarrow{u}L_\infty(\mu)\xrightarrow{i}L_2(\mu)\xrightarrow{v}Y$$

(here i is the canonical inclusion mapping). The smallest such constant C is denoted $\pi_2(T)$.

Suppose that X_0 is a subspace of X; clearly $\pi_2(T|_{X_0})\leqq\pi_2(T)$, which is an observation that is used below.

If X is a finite-dimensional normed space then it follows that, for each $\varepsilon>0$, there exists $m\geqq1$ such that F admits the following factorisation:

$$X\xrightarrow{v}l_\infty^m\xrightarrow{\Delta}l_2^m\xrightarrow{w}Y$$

in which $\|v\|\leqq1$, $\|w\|\leqq1$ and $\|\Delta\|\leqq\pi_2(T)+\varepsilon$, where Δ is the diagonal mapping given by $\Delta(e_i)=\delta_ie_i$ for some $\delta_i\geqq0$ $(1\leqq i\leqq m)$, and $(e_i)_{i=1}^m$ is the standard basis of l_∞^m.

Finally, we recall the notion of type and cotype. A Banach space X is said to be of type 2 if there exists a constant $T>0$ such that for every finite collection of vectors $x_1,\dots,x_n$ in X, we have

$$\left(\int_0^1 \left\| \sum_{i=1}^n r_i(t)x_i \right\|^2 dt\right)^{1/2} \leqq T\left(\sum_{i=1}^n \|x_i\|^2\right)^{1/2},$$

where $(r_i(t))_{i=1}^\infty$ is the sequence of Rademacher functions. The smallest such T, denoted $T_2(X)$, is called the type 2 constant of X. It is easily seen that the above inequality will remain valid if the Rademacher functions are replaced by independent $N(0,1)$ Gaussian random variables, a remark which is used below. X is said to be of cotype 2 if there exists a constant $K>0$ such that for every finite collection of vectors $x_1,\dots,x_n$ in X, we have

$$\left(\int_0^1 \left\| \sum_{i=1}^n r_i(t)x_i \right\|^2 dt\right)^{1/2} \geqq \frac{1}{K}\left(\sum_1^n \|x_i\|^2\right)^{1/2}.$$

The smallest such K, denoted $K_2(X)$, is called the cotype 2 constant of X.

3. The volume ratio

The relation between the volume ratio and the existence of large almost Euclidean subspaces is explained by the following theorem of M. Rogalski ([12]), which is a slight refinement of a result proved by the second-named author ([17], [18]).

THEOREM A. *Suppose that E is an n-dimensional normed space, that* $0<\theta<1$ *and that* $k \leqq \theta n$. *Then, identifying E with* l_2^n *so that* $b_2^n \subset B_E$, *we have for all* $t>1$

$$\mu_k^n\left\{F \in G_k^n : d(F, l_2^k) \leqq \psi(\theta) t^{1/(1-\theta)} \left(\frac{\mathrm{vol}_n(B_E)}{\mathrm{vol}_n(b_2^n)}\right)^{1/n(1-\theta)}\right\} > 1-\frac{1}{t^n},$$

where

$$\psi(\theta) = \left(\frac{4}{\sqrt{3}\theta}\right)^{\theta/(1-\theta)} \frac{1}{1-\theta} \leqq \left(\frac{4}{\sqrt{3}}+1\right)^{1/(1-\theta)}.$$

We now state and prove the following unpublished (besides seminar notes [12]) observation of N. Tomczak-Jaegermann, whom we thank for the permission to include it here. For a slightly more general fact see Remark 3 in §4.

THEOREM B. *Suppose that E is an n-dimensional normed space which has been identified with* l_2^n *so that* b_2^n *is the ellipsoid of maximum volume contained in* B_E. *For each* $1 \leqq k \leqq n$ *and for each k-dimensional subspace F of E, we have*

$$\left(\frac{\mathrm{vol}_k(B_F)}{\mathrm{vol}_k(b_2^k)}\right)^{1/k} \leqq \sqrt{\frac{\dim E}{\dim F}}\, T_2(F^*).$$

In particular, if $\dim F \geqq \theta n$ *then*

$$\left(\frac{\mathrm{vol}_k(B_F)}{\mathrm{vol}_k(b_2^k)}\right)^{1/k} \leqq \sqrt{\frac{1}{\theta}}\, T_2(F^*).$$

The proof of Theorem B makes use of the following inequality of W. Blaschke ([1]) and L. A. Santalo ([15]). We first remark that the standard inner product on l_2^n makes it possible to identify simultaneously an n-dimensional normed space E and its dual E^* with l_2^n.

THEOREM C. *Suppose that E is a finite-dimensional normed space which has been identified with* l_2^n. *Then*

$$\mathrm{vol}_n(B_E)\cdot \mathrm{vol}_n(B_{E^*}) \leqq (\mathrm{vol}_n(b_2^n))^2.$$

A new and interesting proof of Theorem C appears in [14], and some applications to the geometry of Banach spaces are given in [9], [3] and [4]. We also require two further results, which are both known. A short proof of the first of these, which appears in [2], is reproduced here to make our account more self-contained. Proposition E on the other hand is less recondite; a proof can be found in [2].

PROPOSITION D. *Suppose that E is a Banach space and that T is an operator from* l_2^n *into E. Then*

$$l(T) \leqq T_2(E)\pi_2(T^*).$$

PROOF. Let $\varepsilon > 0$ be given, and suppose that

$$E^* \xrightarrow{v} l_\infty^m \xrightarrow{\Delta} l_2^m \xrightarrow{w} l_2^n$$

is a factorisation of T^* with $\|v\| \leqq 1$, $\|w\| \leqq 1$ and $\|\Delta\| \leqq (1+\varepsilon)\pi_2(T^*)$, where $\Delta(e_i) = \delta_i e_i$ $(1 \leqq i \leqq m)$. Then

$$\begin{aligned} l(T) &= l(v^*\Delta^* w^*) \\ &\leqq l(v^*\Delta^*)\|w^*\| \\ &\leqq \left(\int_\Omega \left\|\sum_{i=1}^n g_i(\omega) v^*\Delta^* e_i\right\|^2 dP\right)^{1/2} \end{aligned}$$

(where $g_1, \ldots, g_n$ are independent real gaussian $N(0,1)$ random variables on a probability space (Ω, Σ, P))

$$\leqq T_2(E)\left(\sum_{i=1}^n \|v^*\Delta^* e_i\|^2\right)^{1/2}$$

$$\leqq T_2(E)\left(\sum_{i=1}^{n}\|\Delta^* e_i\|^2\right)^{1/2}$$

$$= T_2(E)\|\Delta^*\|$$

$$\leqq T_2(E)(1+\varepsilon)\pi_2(T^*).$$

Since ε is arbitrary, the result follows.

PROPOSITION E. *Suppose that E is an n-dimensional normed space and that $v : E \to l_2^n$ is an operator such that $v^{-1}(b_2^n)$ is the ellipsoid of maximum volume in B_E. Then $\pi_2(v) = n^{1/2}$.*

PROOF OF THEOREM B ([12]). Suppose that $v : E \to l_2^n$ is an operator of the type described in Proposition E.

Let $w = (\pi \circ v)|_F$, where $\pi : l_2^n \to v(F)$ is an orthogonal projection. The Blaschke–Santalo inequality gives

$$\mathrm{vol}_k(w(B_F))/\mathrm{vol}_k(b_2^k) \leqq \mathrm{vol}_k(b_2^k)/\mathrm{vol}_k(w^{*-1}(B_{F^*}))$$

$$= 1 \Big/ \left(\int_{S^{k-1}} (1/\|w^*(x)\|^k)\,d\sigma_{k-1}\right)$$

$$\leqq \left(\int_{S^{k-1}} \|w^*(x)\|^2\,d\sigma_{k-1}\right)^{k/2}$$

by Jensen's inequality applied to the function $t \to t^{-k/2}$. Applying Propositions D and E, observing that $\pi_2(w) \leqq \pi_2(v)$, we obtain

$$(\mathrm{vol}_k(w(B_F))/\mathrm{vol}_k(b_2^k))^{1/k} \leqq \left(\int_{S^{k-1}} \|w^*(x)\|^2\,d\sigma_{k-1}\right)^{1/2}$$

$$= (1/\sqrt{k})\,l(w^*)$$

$$\leqq (1/\sqrt{k})\,\pi_2(w)\,T_2(F^*)$$

$$\leqq \sqrt{(n/k)}\,T_2(F^*).$$

The statement of Theorem B now follows at once.

4. The Kashin decomposition for Banach spaces of cotype 2

The other main ingredient in the argument which we present below is the following theorem of G. Pisier ([11]).

THEOREM F. *There exists an absolute constant $C>0$ such that for every n-dimensional normed space E, we have*

$$T_2(E^*)\leqq C\log(d(E,l_2^n)+1)\cdot K_2(E).$$

Combining Theorems A, B and F, we are now in a position to prove a theorem of the type described in the Introduction. The proof utilises ideas which V. D. Milman has used in [9] to prove the existence of large almost Euclidean quotient spaces of subspaces in an arbitrary finite-dimensional normed space (see also Remark 4). To avoid unnecessary repetition it is to be understood that $C_1, C_2, \dots$ are absolute constants.

THEOREM 1. *Suppose that E is an n-dimensional normed space which has been identified with l_2^n so that b_2^n is the ellipsoid of maximum value contained in B_E, that $0<\theta<1$ and that $k=[\theta n]$. Then*

$$\mu_k^n\left\{F\in G_k^n: d(F,l_2^k)\leqq \max\left\{(CK_2(E))^{7/(1-\theta)}, \left(\frac{7}{1-\theta}\right)^{7/(1-\theta)}\right\}\right\}\geqq 1-\frac{1}{2^{k-1}}$$

where C is an absolute constant.

For the proof of Theorem 1 we need the following lemma, whose proof we postpone until §5.

LEMMA 1. *Given $\alpha>1$ let $g:[0,1]\to[1,\infty)$ satisfy*
(i) $g(\theta x)\leqq(\alpha\log g(x))^{1/(1-\theta)}$
for $x\in[1/2,1]$ and $\theta\in[0,1)$. Then, for all $x\in[0,1)$

(ii) $g(x)<\max\left\{\alpha^{7/(1-x)}, \left(\frac{7}{1-x}\right)^{7/(1-x)}\right\}$.

PROOF OF THEOREM 1. Suppose that F is a k-dimensional subspace of E. Given $\theta\in(0,1)$, it will be convenient to consider subspaces H of F of dimension $s=[\theta k]+1$ rather than the dimension $[\theta k]$, which is given by applying directly Theorem A. But since every subspace of codimension one is $(2+\varepsilon)$-complemented, it follows that the corresponding estimate for the Banach–Mazur distance is worse than in Theorem A only by a constant factor ($\leqq 3$; one can easily show that if $F_0\subset F$, $\dim F/F_0=1$, then $d(F,l_2^s)\leqq 2d(F_0,l_2^{s-1})+1$), while the estimate for the measure is unchanged. So, for all $F\subset E$, $\theta\in(0,1)$ and $t>1$,

$$\mu_s^F\left\{H\in G_s^F: d(H,l_2^s)\leqq\left(C_1t\left(\frac{\mathrm{vol}_k(B_F)}{\mathrm{vol}_k(b_2^k)}\right)^{1/k}\right)^{1/(1-\theta)}\right\}>1-\frac{1}{t^k}. \tag{1}$$

For $x\in[0,1]$, define

$$g(x)=\inf\left\{\lambda : \exists k \geqq nx,\ \mu_k^n\{F\in G_k^n : d(F, l_2^k)\leqq \lambda\}\geqq 1-\frac{1}{2^{k-1}}\right\}.$$

Notice that if $d(F, l_2^k)\leqq\lambda$, then, by Theorems B and F,

$$\left(\frac{\mathrm{vol}_k(B_F)}{\mathrm{vol}_k(b_2^k)}\right)^{1/k}\leqq C_2(x)K_2(E)\log\lambda,\qquad \text{where } C_2(x)\leqq\frac{C_3}{\sqrt{x}}; \tag{2}$$

in particular $C_2(x)\leqq C_3\sqrt{2}=C_4$ if $x\geqq 1/2$.

We claim that g satisfies the assumptions of Lemma 1 with $\alpha = 2C_1C_4K_2(E)$, where C_1 is the constant appearing in (1). Indeed, if $x\in[1/2,1]$ and $\theta\in[1/2,1)$, then applying (1) with $t=2$ and (2) with $\lambda = g(x)$ we see that if $d(F, l_2^k)\leqq g(x)=\lambda$, then

$$\mu_s^F\{H\in G_s^F : d(H, l_2^s)\leqq(2C_1C_4K_2(E)\log\lambda)^{1/(1-\theta)}=(\alpha\log g(x))^{1/(1-\theta)}\}>1-\frac{1}{2^s}, \tag{3}$$

where $s=[\theta k]+1>\theta k\geqq\theta xn$. Now, using (0) from §2 and the definition of g we obtain

$$\begin{aligned}\mu_s^n\{H\in G_s^n : d(H, l_2^s)\leqq(\alpha\log g(x))^{1/(1-\theta)}\}&\geqq\mu_k^n\{F : d(F, l_2^k)\leqq g(x)\}\cdot(1-1/2^s)\\&\geqq(1-1/2^{k-1})(1-1/2^s).\end{aligned}$$

Now if $\theta<1-1/k$, and consequently $s<k$, the latter quantity exceeds $1-1/2^{s-1}$ and so $g(\theta x)\leqq(\alpha\log g(x))^{1/(1-\theta)}$. This is also trivially true if $\theta\geqq 1-1/k$ and $s=k$ (at least as long as $\alpha\log g(x)\geqq 5/4$, which we can assume). This shows that g satisfies the assumptions of Lemma 1 with $\alpha = C_5K_2(E)$ and so

$$g(x)\leqq\max\left\{(C_5K_2(E))^{7/(1-x)},\left(\frac{7}{1-x}\right)^{7/(1-x)}\right\},$$

which is exactly the assertion of Theorem 1.

REMARK 1. The statement of Theorem 1 is not intended to be optimal. In particular, an examination of the proof reveals that it is possible to improve the index $7/(1-\theta)$ to $(2+\varepsilon)/(1-\theta)$ in the first exponent and to $(1+\varepsilon)/(1-\theta)$ in the second term (both in the base and in the exponent) for any $\varepsilon>0$ provided that the measure estimate is suitably modified.

We now obtain the theorem stated in the abstract with an estimate for the Banach–Mazur distance from Euclidean space of the subspaces in the decomposition.

THEOREM 2. *There exists an absolute constant $C>0$ such that every $2n$-*

dimensional normed space E contains an n-dimensional subspace F with the following properties:

(i) $d(F, l_2^n) \leqq CK_2(E)^2(\log(K_2(E)+1))^4$;

(ii) $d(F^\perp, l_2^n) \leqq CK_2(E)^2(\log(K_2(E)+1))^4$

where $F^\perp$ is the orthogonal complement of F with respect to the inner product induced by the John ellipsoid of E.

PROOF OF THEOREM 2. In general "decomposition" results follow immediately from the "measure theoretic" results such as Theorem 1. This is also the case here, but instead of (i) and (ii) we get estimates of the Banach–Mazur distances of $CK_2(E)^{14}$. To get (i) and (ii) we must work a bit harder. Setting $1-\theta = 1/\log(K_2(E))$ in Theorem 1 gives a set of large measure of subspaces G of dimension $k = [2\theta n]$ such that

$$d(G, l_2^k) \leqq (C_6 K_2(E))^{4\log K_2(E)}.$$

We now apply equation (3) in Theorem 1 with

$$\phi = 1/2\theta \quad \text{and} \quad \lambda = (C_6 K_2(E))^{4\log K_2(E)}.$$

This gives rise to a set of n-dimensional subspaces F satisfying

$$\begin{aligned} d(F, l_2^n) &\leqq (C_6 K_2(E)(\log K_2(E))^2)^{1/(1-\phi)} \\ &\leqq CK_2(E)^2(\log(K_2(E)+1))^4. \end{aligned}$$

Since this set is of large measure (in particular, of measure $>\frac{1}{2}$), there exists a member of the set whose orthogonal complement is also a member; the latter remark follows from the fact that the mapping $F \to F^\perp$ is a measure preserving transformation of G_n^{2n}. This completes the proof of the theorem.

REMARK 2. The argument of Theorem 2 shows that for each θ in $(0,1)$ there exists a constant $C(\theta)$ such that every n-dimensional normed space E contains a subspace F of dimension $k = [\theta n]$ (in fact, a set of large measure of such subspaces) with

$$d(F, l_2^k) \leqq (C(\theta)K_2(E)(\log(K_2(E)+1))^2)^{1/(1-\theta)}$$

Moreover, the constant $C(\theta)$ may be chosen to be uniformly bounded on any interval $0 < \theta \leqq \theta_0 < 1$. At least for small values of θ this improves the dependence on $K_2(E)$ given by Theorem 1.

REMARK 3. Theorems B, 1 and 2 remain true if we replace the inner product norm on E generated by the John ellipsoid by any norm $|\cdot|$ satisfying:

$$\|\cdot\|_E \leqq |\cdot|, \quad \pi_2(I:(E,\|\cdot\|)\rightarrow(E,|\cdot|))\leqq a\sqrt{n};$$

the constants "C" in Theorems 1 and 2 depend then additionally on a and we have to introduce the factor "a" in the assertion of Theorem B.

Theorem 2 sheds light on the local structure of Banach spaces of cotype 2. The following result is an immediate consequence.

COROLLARY 3. *Suppose that X is a Banach space of cotype 2. All of the finite-dimensional subspaces of X admit a Kashin decomposition with a uniform bound on the Banach–Mazur distance from Euclidean space of the members of the decomposition.*

We conclude the discussion with some remarks about the relationship between cotype and the Kashin decomposition. First, it follows from the results of [5] and [8] that any Banach space which satisfies the conclusion of Corollary 3 is necessarily of cotype $2+\varepsilon$ for each $\varepsilon>0$. Secondly, it is shown in [19] that there exists a Banach space X which satisfies the conclusion of Corollary 3 and yet is not of cotype 2. Lastly, there is an example in [5] of a Banach space of cotype $2+\varepsilon$ for each $\varepsilon>0$ which does not satisfy the conclusion of Corollary 3. Thus Corollary 3 is in a qualitative sense the best result. We now present an example which shows quantitatively the optimality of Corollary 3. Suppose that X is a Banach space and that $n\geqq 1$. Then $K_{2,n}(X)$ is defined to be the least number K such that for every collection of n vectors $x_1,\dots,x_n$ in X, we have

$$\left(\int_0^1\left\|\sum_{i=1}^n x_i r_i(t)\right\|^2 dt\right)^{1/2}\geqq\frac{1}{K}\left(\sum_{i=1}^n\|x_i\|^2\right)^{1/2}.$$

Thus X is of cotype 2 if and only if $\sup\{K_{2,n}(X): n\geqq 1\}<\infty$. Moreover, it follows from the results of [20] and Theorem 2 that if $K_{2,n}(X)\leqq C$ then every n-dimensional subspace of X admits a decomposition with distances depending only on C.

EXAMPLE 4. Suppose that $(\lambda_n)_{n\geqq 1}$ is any unbounded increasing sequence of positive numbers. There exists a Banach space X such that $K_{2,n}(X)=O(\lambda_n)$ and X does not satisfy the conclusion of Corollary 3.

IDEA OF THE PROOF. We may assume that $1\leqq\lambda_n\leqq n^{1/4}$. For each $n\geqq 2$, let $q(n)$ be defined by $n^{1/2-1/q(n)}=\min\{\lambda_n, n^{1/2-1/q(n-1)}\}$ so that $2\leqq q(n)\leqq q(n-1)\leqq 4$ for all $n\geqq 2$. Let $E_n=l^n_{q(n)}$ and let $X=(\sum_{n=2}^\infty\oplus E_n)_2$. An elementary argument using some results from [7] shows then that $K_{2,m}(X)\leqq\lambda_m$ for all m, while, by Example 3.1, [5], the E_n's do not contain half-dimensional "nearly Hilbertian" subspaces.

Example 4 shows that Theorem 2 is optimal in the sense that there is no weaker assumption than uniform boundedness on the constants $K_{2,n}(X)$ which is still a sufficient condition for the existence of Kashin decompositions.

Finally, we restate a question which has been stated many times before (e.g. [10]).

PROBLEM. Does there exist a function f such that $\mathrm{vr}(E) \leqq f(K_2(E))$ for every finite-dimensional normed space E?

5. Proof of Lemma 1, and remarks

Lemma 1 will follow immediately from

LEMMA 2. *Given $K > 0$ let $f : [0,1] \to \mathbf{R}^+$ satisfy*

(j) $f(\theta x) \leqq (\log f(x) + K)/(1-\theta)$ *for* $x \in [1/2, 1]$ *and* $\theta \in [0,1)$.

Then for all $x \in [0,1)$,

(jj) $f(x) < \dfrac{7}{1-x} \max\left\{K, \log \dfrac{7}{1-x}\right\}$.

Indeed, to get Lemma 1 from Lemma 2, set $f = \log g$ and $K = \log \alpha$.

PROOF OF LEMMA 2. Assume first that f is nondecreasing, in particular $f(x) \leqq f(1)$ for $x \in [0,1]$. Clearly (jj) is then satisfied if x is sufficiently close to 1. Consider the smallest number $\sigma \in [0,1)$ such that (jj) is satisfied for $x \in (\sigma, 1)$. It is enough to show that $\sigma = 0$. To this end, suppose $\sigma > 0$ and let $\theta = \min\{\sqrt{\sigma}, 2\sigma\}$, $x = \sigma/\theta$; observe that then

$$\frac{1}{1-x} < \frac{2}{1-\sigma} \quad \text{and} \quad \frac{1}{1-\theta} < \frac{2}{1-\sigma}, \qquad x \in (\sigma, 1) \cap [1/2, 1).$$

Thus applying (j) for this choice of x, θ we get

$$\begin{aligned} f(\sigma) < \frac{\log f(x) + K}{1-\theta} &< \frac{1}{1-\theta}\left[\log\left(\frac{7}{1-x}\max\left\{K, \log\frac{7}{1-x}\right\}\right) + K\right] \\ &< \frac{2}{1-\sigma}\left[\log\left(\frac{14}{1-\sigma}\max\left\{K, \log\frac{14}{1-\sigma}\right\}\right) + K\right] \\ &= \frac{2}{1-\sigma}\left[\log\frac{14}{1-\sigma} + K + \log\left(\max\left\{K, \log\frac{14}{1-\sigma}\right\}\right)\right]. \end{aligned}$$

Now if $K > \log(14/(1-\sigma))$, then we get from the above

$$f(\sigma) < \frac{2}{1-\sigma}[K + K + \log K] < \frac{2}{1-\sigma} \cdot \frac{5}{2} K < \frac{7K}{1-\sigma}.$$

If, on the other hand, $K \leqq \log(14/(1-\sigma))$, then similarly

$$f(\sigma) < \frac{2}{1-\sigma}\left[\log\frac{14}{1-\sigma} + \log\frac{14}{1-\sigma} + \log\log\frac{14}{1-\sigma}\right]$$
$$< \frac{5}{1-\sigma}\log\frac{14}{1-\sigma}$$
$$< \frac{7}{1-\sigma}\log\frac{7}{1-\sigma},$$

the last inequality following from the fact that $\log 2u < 7/5 \log u$ if $u > 7$. Therefore in either case (jj) is satisfied for $x = \sigma$ and hence (as f is nondecreasing and the right-hand side of (jj) continuous) also in some interval $(\sigma - \varepsilon, \sigma)$. This would contradict the minimality of σ and so we must have $\sigma = 0$.

To settle the case of general f, define, for $\tau < 1$, $f_\tau(x) = \sup_{s \leqq \tau x} f(s)$. Since by (j),

$$f(s) \leqq \frac{\log f(1) + K}{1-s},$$

each f_τ is finite (and also clearly nondecreasing). Thus (jj) holds if we replace f by f_τ; letting $\tau \to 1$ concludes the argument.

REMARK 4. Arguing almost identically as in Lemmas 1 and 2 one can show that if $h : [0,1] \to [1,\infty)$ satisfies

$$h(\theta x) \leqq (C \log h(x))^{1/(1-\theta)^2} \qquad \text{for } \theta \in [0,1) \quad \text{and} \quad x \in [0,1],$$

then

$$h(x) \leqq \max\left\{c^{15/(1-x)^2}, \left(\frac{15}{(1-x)^2}\right)^{15/(1-x)^2}\right\}.$$

Combing this with formula (5) from V. D. Milman [9] shows that, given $d > 1$, every n-dimensional normed space has a quotient of a subspace, say F, of dimension $k \geqq \theta n$ with $d(F, l_2^k) \leqq d$ and

$$\theta = 1 - c' \sqrt{\frac{\log\log d}{\log d}}$$

(just let $h(x) = \inf d(F, l_2^{\dim F})$ where the infimum is taken over all F's which are quotients of a subspace of E with $\dim F \geqq nx$). This improves the estimate from [9]. We do not know, however, any "measure theoretic" version of that result. Let us state the following.

PROBLEM. Do there exist $C<\infty$ and $\tau\in(0,1)$ (resp. a function on $(0,1):\tau\to C=C(\tau)$) such that every n-dimensional normed space E admits Euclidean structure for which if $F\subset G$ are fixed subspaces of E with $\dim F=[1/2\,\theta n]$ and $\dim G=[\theta n]$, then $g_{UF}(UG^{\perp})$ is C-Hilbertian for "most of" $U\in O(n)$ (by g_H we denote the quotient map $E\to E/H$). In particular, does every n-dimensional space admit a quotient, whose cotype 2 constant is bounded by an absolute constant (resp. "most of" quotients). The same question for volume ratio.

REMARK 5. One can weaken significantly the condition (j) of Lemma 2 and still get that $f(x)<F(x)$, where F is some universal function defined on $[0,1)$. For example, one can replace (j) by

(j)′ $$f(\theta x)\leqq\phi(\theta)\psi(f(x))$$

where ϕ and ψ are nonnegative, nondecreasing functions (on $(0,1)$ and $(0,\infty)$ respectively) with $\phi(z)=O(z^p)$ for some $p<1$,

$$\phi(\theta)=O\left(\left(\frac{1}{1-\theta}\right)^q\right)\qquad\text{for some } q<\infty$$

and still get

(jj)′ $$f(x)<C\left(\frac{1}{1-\theta}\right)^{q/(1-p)}$$

(with C depending on p, q and the constants involved) in estimates of the order of growth of ϕ and ψ.

REMARK 6. Let us return to the formula (0) from §2. Notice the following consequence of it: if $1\leqq k<l<n$ and $A\subset G_l^n$, then

$$\gamma_n(A)\overset{\text{df}}{=}\mu_k^n\{H\in G_k^n : H\subset F \text{ for some } F\in A\}\geqq\mu_l^n(A).$$

However, this estimate is apparently not the best possible. We have the following "isoperimetric"

PROBLEM. What is $\inf\{\gamma_k(A):\mu_l^n(A)=\delta\}$ for $\delta\in(0,1)$?

If $k=1$, $l=2$ and $n=3$, then we conjecture that the infimum equals $\sqrt{\delta}(2-\delta)$ ($>\delta$ for any $\delta\in(0,1)$). One can also ask a more general question, which is more directly related to our argument: what is $\inf\mu_k^n(B)$ over all $B\subset G_k^n$ such that $\mu_l^n\{F\in G_l^n:\mu_k^F(B\cap G_k^F)\geqq\beta\}\geqq\delta$? (If $\beta=1$, we get the problem above.)

REMARK 7. Suppose that P is a symmetric convex polytope in $\mathbf{R}^n$ with $2s$ vertices and that E is a normed space whose unit ball is affinely equivalent to P. From Theorem B we obtain

$$\mathrm{vr}(E) \leqq T_2(E^*) \leqq T_2(l_\infty^s) \leqq c\sqrt{\log s}$$

(see e.g. [10]). So the ellipsoid $\mathscr{E}$ of maximum volume in P satisfies

$$\left(\frac{\mathrm{vol}_n(P)}{\mathrm{vol}_n(\mathscr{E})}\right)^{1/n} \leqq c\sqrt{\log s}.$$

This should be compared with the estimate

$$\left(\frac{\mathrm{vol}_n(P)}{\mathrm{vol}_n(\mathscr{E})}\right)^{1/n} \leqq c\left(\frac{s}{n}\right)$$

(e.g. [17]), which is superior when s and n are of similar magnitude. We would conjecture that we actually have

$$\left(\frac{\mathrm{vol}_n(P)}{\mathrm{vol}_n(\mathscr{E})}\right)^{1/n} \leqq c\sqrt{\log\left(\frac{s}{n}\right)}.$$

NOTE. After this paper was written we discovered that V. D. Milman has obtained some superior estimates for the distance from Euclidean space. He has proved that, given θ in $(0,1)$, every n-dimensional normed space E contains a subspace F with $\dim F = k \geqq \theta n$ and

$$d(F, l_2^k) \leqq c(\theta) K_2(E) \log(K_2(E)+1).$$

Added in proof. The problem stated at the end of §4 has recently been solved by J. Bourgain and V. D. Milman in their preprint *Sections Euclidiennes et volume des corps symétriques convèxes dans* $\mathbf{R}^n$.

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